Contributions

Omer Levy*, Rann Smorodinsky and Moshe Tennenholtz

Undivide and Conquer: On Selling a Divisible and Homogeneous Good

Abstract: With the prevalence of cloud computing emerges the challenges of pricing cloud computing services. There are various characteristics of cloud computing which make the problem unique. We study an abstract model which focuses on one such aspect – the sale of a homogeneous and fully divisible good. We cast onto our model the idea of bundling, studied within the context of monopolist pricing of indivisible goods. We demonstrate how selling a divisible good as an indivisible one may increase seller revenues and characterize when this phenomenon occurs, and the corresponding gain factors.

Keywords: bundling, VCG, auctions

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1 Introduction

The software industry has undergone tremendous changes with the introduction of-*cloud services*. Initially various applications were offered as cloud services in what has become commercially known as the-*SaaS* (Software as a Service) business model. Nowadays, not only applications but also computational infrastructure such as computing power (CPUs) and memory (disk space) are offered in a similar service model (sometimes dubbed-*IaaS*) by companies such as Amazon, Microsoft, many telecommunication companies and more. A recent

***Corresponding author: Omer Levy,** Department of Computer Science, Bar-Ilan University, Ramat Gan, Israel, E-mail: omerlevy@gmail.com
Rann Smorodinsky, Department of Industrial Engineering and Management, Technion – Israel Institute of Technology, Haifa, Israel, E-mail: rann@ie.technion.ac.il
Moshe Tennenholtz, Department of Industrial Engineering and Management, Technion – Israel Institute of Technology, Haifa, Israel; Microsoft Research, Herzliya, Israel, E-mail: moshet@microsoft.com

Gartner report provides some insight into the market size (Pettey and van der Meulen 2012):

> The public cloud services market is forecast to grow 19.6 percent in 2012 to total $109 billion worldwide, according to Gartner, Inc. Business process services (also known as business process as a service, or BPaaS) represent the largest segment, accounting for about 77 percent of the total market, while infrastructure as a service (IaaS) is the fastest-growing segment of the public cloud services market and is expected to grow 45.4 percent in 2012.

The market for cloud services, and in particular IaaS, has a variety of unique features which make it different from other markets. One particular aspect is that the goods themselves (memory and CPU) are homogeneous and fully divisible. Thus, the pricing schemes that could prevail may also be unique to this market. One of the interesting and non-intuitive conclusions from the study of optimal pricing by monopolists is that bundling goods together may yield higher revenues (Palfrey 1983; Ghosh, Nazerzadeh, and Sundararajan 2007). We cast this to the scenario of pricing a homogeneous and fully divisible good and ask the following simple question – when is it better for the seller to sell such a good as a whole rather than as a divisible good? Our first example[1] demonstrates that this question is indeed not trivial:

Example 1. *Assume a single divisible and homogeneous resource, such as disk-space, is sold to two buyers using the standard VCG mechanism* (Vickrey 1961; Clarke 1971; Groves 1973). *Each buyer, i, has a private valuation function,* $v_i : [0, 1] \to \mathbb{R}_+$, *associating a valuation to any fraction of the good. Let* $v_1(x) = v_2(x) = 5 + x$, *for* $0.5 \le x \le 1$, *and 0 otherwise. Now compare between a seller that allows bidders to express their entire valuation function and a seller that only allows customers to express the value of the good as a whole (i.e., considers the good indivisible). It is easy to verify that truthfulness is a dominant strategy in both cases.*

Note that in the first case the seller will end up giving each buyer half of the good (this is the most efficient outcome) but will only charge the utility loss incurred on the other buyer, which is $6 - 5.5 = 0.5$. *Hence, the resulting revenue for the first seller will be 1. In the second case, however, the seller will allocate the good to one of the players and will price him according to the utility loss of the other player which is* $6 - 0 = 6$.

1 As will be evident in the following sections, this example does not meet the standards of sensible utility functions, such as smoothness and nullification at zero. We chose this example because it is numerically elegant and simple to illustrate. It is also possible to reconstruct a similar scenario with $v_1(x) = v_2(x) = \sqrt{x}$ and $v_3(x) = 2x$ that exemplifies the superiority of each mechanism.

Now consider adding a third buyer with the valuation function $v_3(x) = 10x,\ 0.5 \leq x \leq 1$, *and 0 otherwise. Now the second seller allocates the whole good to buyer 3 and extracts a revenue of 6, whereas the first seller allocates the good equally between customers 1 and 2 and each pay 4.5. Thus, the phenomenon is reversed and the first seller is advantageous.*

In our work, we mainly compare two selling schemes of a homogeneous good. In the first scheme the good is divided up among various buyers while in the second it is sold as a whole. In particular, we provide (almost) tight conditions for the latter option to yield higher revenues. We go on to compute bounds for the related gains. That is, we compute bounds on the ratio of revenues between the two schemes. Following the literature in computer science dealing with bundling (Ghosh, Nazerzadeh, and Sundararajan 2007) and in particular the literature on bundling equilibrium (Holzman et al. 2004), we focus on the use of VCG mechanisms. Truthfulness of such VCG mechanisms allows for a relatively straightforward analysis, circumventing challenges that emerge in the analysis of equilibria of other mechanisms, such as buyers' beliefs over other players' valuations or any common prior assumptions on the distribution of valuations.

1.1 Related literature

The work we do is clearly related to monopolist pricing and bundling on the one hand but also to the algorithmic work done on cloud services pricing on the other hand. We describe this related work briefly.

1.1.1 Pricing and bundling

Adams and Yellen (1976) introduce a model of a monopolist with two types of goods and demonstrate scenarios, with complete information, where the monopolist is better off bundling these two goods. Palfrey (1983) extends these ideas to a monopolist model with a richer portfolio of products and incomplete information about buyers' valuations. He finds that when the number of buyers is small it may be worthwhile for the monopolist to bundle all the goods ("pure bundling"), which will consequently generate some inefficiency. This inefficiency will typically vanish as the number of buyers increases.

Jehiel, Meyer-Ter-Vehn, and Moldovanu (2007) study auctions with "mixed bundles" and show that they are often revenue superior to no bundling and to pure bundling. Ghosh, Nazerzadeh, and Sundararajan (2007) explore the same question in the context of the digital advertising market. Whereas the aforementioned works explore a discrete model of indivisible heterogeneous goods, we analyze the same phenomenon for the continuous and homogeneous case. Similar to Palfrey (1983), we only focus on the “pure bundling” option and defer mixed bundles to future work.

Most similar to our work, Smirnov and Wait (2011) also study bundling in the context of a single homogeneous and divisible good. They consider buyers which have a similar valuation function in the sense that for any two buyers, the ratio of the valuation from any share of the good is a constant. This, in particular, suggests a natural order on the buyers. In our work we do not consider such restrictions but nevertheless reach similar conclusions. In particular when the valuation functions are all convex (super-additive) then bundling is always revenue superior to non-bundling. However, whenever valuation functions are all concave (sub-additive) the non-bundled option may become more attractive as the number of buyers increases. In addition, we go beyond the ordinal analysis of Smirnov and Wait (2011) and provide a cardinal quantitative analysis of the potential gain or loss from bundling.

The approach taken in this paper partially relates to recent attempts of introducing strategic equilibrium consideration into cake-cutting (Brânzei and Bro Miltersen 2013) and to mechanism design approaches of dividing a homogeneous good (Feige and Tennenholtz 2011). However, our aim is to consider revenue maximization in fundamental economic-monetary contexts using classical mechanism by deciding on whether to *un*-divide, rather than considering fairness and social welfare consideration in settings where no money is available.

1.1.2 Cloud pricing

The efficient allocation of computational resources, and in particular bandwidth, has also been studied by the distributed computing community. Specifically, the implementation of resource distribution through auctions has already been introduced in Kelly (1997) and Sanghavi and Hajek (2004). Due to the obvious communication constraints on these systems, most authors have imposed ad hoc limitations on the level of detail players (buyers) can provide the auction about their own utility function (in fact, in some papers, such limitations were imposed on the underlying valuation functions, which of course affects the actual message space). One such natural limitation is the POINT mechanism, where buyers can only bid on a particular quantity (e.g., Lazar and Semret 1999). We briefly discuss the POINT mechanism at the end of this paper and show that it is not trivial to determine if and when it dominates other mechanisms. A more general version of POINT, which allows a finite (constant) number of quantities, was later studied as well by Maillé and Tuffin (2004).

Recent work by Ben-Yehuda et al. tries (2012) to accommodate the dynamic nature of the IaaS business model where resources such as CPU and memory are dynamically allocated in an ever-changing environment. While these works are motivated by building actual working systems for resource allocation, and constantly attempt to (empirically) do better than the benchmark of existing systems, our work provides a theoretical analysis of comparing two methods for auctioning such resources.

1.1.3 Systems research

The study of allocating divisible goods, often in the context of resources such as CPU and memory, is also undertaken by the systems research community. In fact, part of this work utilizes concepts and methodologies borrowed from the game theoretic literature. The lion's share of this body of work focuses on allocation mechanisms which induce a *fair* allocation of the resources. For example, Ghodsi et al. (2011) suggest a strategy-proof mechanism for fairly allocating multiple types of resource in a datacenter. In contrast, our work ignores questions of fairness and but rather focuses on seller's revenue from the allocation.

1.1.4 Communication complexity

It has been suggested that bundling serves to increase a monopolist profit but it is also a tool for designing plausible mechanisms and auctions. Consider an auctioneer that has a large portfolio of goods and buyers which have an arbitrary valuation function over the various subsets of these goods. Typically it would require buyers to list the valuations on all such subsets unless very restrictive valuation functions are considered (e.g. additive ones). This becomes infeasible due to communication constraints when the number of goods is quite small (e.g., for 50 goods it would require transmitting 2^{50} valuations). By bundling the goods together, one can decrease the communication complexity underlying an auction. This motivation is at the heart of work by Holzman et al. (2004) and Holzman and Monderer (2004). In our continuous setting, a similar consideration applies, although it does not motivate our work. By selling the good as a whole, buyers are only required to report their valuation for the good as a whole (one number) as opposed to a full description of the valuation function when full divisibility is allowed.

1.2 Our contribution

Our work introduces the first quantitative analysis of the most basic setting of selling divisible goods through bundling – selling a divisible good as an indivisible unit. This problem is highly motivated by the cloud computing scenario, but more generally complements classical work in economic mechanism design.

2 The model

We consider a setting of a monopolistic seller with a single homogeneous divisible good and a set of n buyers. Each buyer, i, has a valuation function $v_i : [0,1] \to \mathbb{R}_+$, where $v_i(x)$ denotes the value of a fraction x of the good to buyer i. Buyers are risk neutral and have quasi-linear utility functions. The utility of buyer i from obtaining a fraction x of the good at the price p is $u_i(x,p) = v_i(x) - p$. Throughout we assume $v_i \in V$ where V is the set of non-decreasing, continuously differentiable functions, $v_i : [0,1] \to \mathbb{R}_+$, satisfying $v(0) = 0$. For notational convenience we shall assume that $i < j$ implies $v_i(1) \geq v_j(1)$. This assumption will neither be used for the mechanisms we analyze nor-be exploited by the players and hence it is made without loss of generality.

A mechanism is a triplet (M, X, P) composed of a set of messages M and a pair of functions:

1. The allocation function, $X : M^n \to [0,1]^n$, which satisfies $\sum_i X_i(\vec{m}) \leq 1$ for every $\vec{m} \in M^n$.
2. The pricing function $P : M^n \to \mathbb{R}^n$.

Given a vector of messages $\vec{m} = (m_1, \ldots, m_n)$, player (buyer) i will be given the fraction $X_i(\vec{m})$ of the good at the price $P_i(\vec{m})$.

Recall that a VCG mechanism allocates the goods in a way that maximizes social welfare (which is the sum of buyers valuations), and the price each buyer pays is equal the toll it induces on its fellow buyers. We focus on the VCG mechanism for two reasons. First, this allows us to be consistent with the literature on bundling in Computer Science, specifically the literature on bundling equilibrium, and second, due to the unique properties of this mechanism, namely its truthfulness and economic efficiency in allocating the good. More formally, we compare the following two mechanisms:

AON (All or Nothing) Each player is asked to submit the value of getting the whole good, $v_i(1)$ and the mechanism is the classical second price auction for the whole good. Formally, in the mechanism AON the message space is $M=[0,1]$. The allocation is $X_i(\vec{m})=1$ if $\{m_i > m_j | \forall j < i\} \wedge \{m_i \geq m_j | \forall j > i\}$ and $X_i(\vec{m})=0$ otherwise. The payment scheme is $P_i(\vec{m}) = max_{j \neq i} m_j$ if $X_i(\vec{m})=1$ and $P_i(\vec{m})=0$ otherwise. This is simply selling the whole good in a second price action.

FULL Each player is asked to submit a non-decreasing, continuously differentiable, valuation function with no value for zero allocation ($v_i(0)=0$). The seller splits the good among the player in a way that maximizes the social welfare. Prices may be different across players and are the respective VCG prices. Formally, the message space M is $\mathscr{V}$. Let $\{z_i\}_{i=1}^n$ denote the fraction given to buyer i in the social welfare maximizing allocation (in case of more than one such optimal allocation, choose the first in lexicographical order). Thus, $\{z_i\}_{i=1}^n$ maximizes the sum of the players' valuations. Let $\{z_i^j\}_{i \neq j}$ denote the vector of allocations of the optimal allocation when player j is excluded. Then $X_i(\vec{v}) = z_i$ and $P_i(\vec{v}) = \sum_{j \neq i}(v_j(z_j^i) - v_j(z_j))$.

Both mechanisms are truthful. In other words, it is a dominant strategy for player i to report the value $v_i(1)$ in the mechanism AON and to report the function v_i in the mechanism FULL. Consequently, when AON is used the revenue to the seller is the second highest valuation for the whole good, that is $R^{\text{AON}} = R_n^{\text{AON}} = v_2(1)$. Whenever FULL is used the revenue to the seller is the following technical term: $R^{\text{FULL}} = R_n^{\text{FULL}} = \sum_{j=1}^n \sum_{i \neq j}(v_i(z_i^j) - v_i(z_i))$ (we often omit the index n, referring to the number of buyers).

3 Results

In this section, we study and compare the revenue outcome from the two auctions formats: FULL and AON. Our results determine when one format is advantageous over the other from the seller's perspective (revenue-wise). In addition, we provide bounds on the profit ratio from using the optimal option over the inferior one. Assuming no restrictions on the economy, we fall short from providing necessary and sufficient conditions. However, when we consider restricted economies, such as replica economies, our analysis is tight.

The results in this section will be presented without proofs, which are available in the appendix.

3.1 Preliminaries

Consider an optimal allocation of a divisible good among n buyers, and let z_i denote the fraction given to buyer i in this allocation. If the valuations function are differentiable, then first-order conditions imply that the derivative of the function v_i at the allocated amount to player i must have equal values for all players with a positive allocation. Formally, for some α, if player i is allocated the amount $z_i > 0$, then $v_{i'}(z_i) = \alpha$. Let β_j denote this constant derivative for the optimal allocation that excludes player j. We capture these definitions and some additional preliminaries in Proposition 1:

Proposition 1. *Let z_i denote the fraction given to buyer i in the social welfare maximizing allocation of the mechanism under FULL, and similarly, let $\{z_i^j\}_{i \neq j}$ denote the vector of optimal allocations for the set of all players, excluding player j. Then*:

1. $\sum_{i=1}^{n} z_i = 1$.
2. $z_i > 0$ *implies* $v_j'(z_j) \leq v_i'(z_i)$. *In other words, there exists some α such that* $z_i > 0$ *implies* $v_i'(z_i) = \alpha$ *and* $z_i = 0$ *implies* $v_i'(z_i) < \alpha$.
3. *Either* $z_1 = 1$ *or* $\sum_{\{i:z_i>0\}} v_i''(z_i) < 0$.
4. *For all j*, $\sum_{i \neq j} z_i^j = 1$.
5. *There exists some β_j such that* $z_i^j > 0$ *implies* $v_i'(z_i^j) = \beta_j$ *and* $z_i^j = 0$ *implies* $v_i'(z_i^j) < \beta_j$.

3.2 Convex utility functions

In most of the section, our focus will be economies with strictly concave utility functions. However, our first observation is about the opposite case, namely economies with convex valuation functions. We observe that in such cases both FULL and AON generate the same outcome:

Theorem 1. *If all the functions, v_i, are convex then both in AON and in FULL player 1 is allocated the whole good and he pays the valuation of player 2 for the whole good, namely $v_2(1)$.*

3.3 Concave economies – the general case

Hereinafter we assume that all valuation functions are strictly concave. Technically, this means that $v''(x) < 0$. In particular, this yields the following lower and upper bounds for the revenue of the FULL mechanism:

Lemma 1. *Given an economy with strictly concave valuations, the revenue extracted in the FULL mechanism, R^{FULL}, is bounded as follows*:

$$\sum_{j=1}^{n} \beta_j z_j \leq R^{\text{FULL}} \leq \alpha$$

Using this lemma we can derive sufficient conditions for the superiority of FULL over AON as well as sufficient conditions for the opposite to hold:

Corollary 1. *Given an economy with strictly concave valuations, the revenue from the AON mechanism is superior to that of the FULL mechanism whenever $\alpha \leq v_2(1)$. The opposite holds if $\beta_j \geq v_2(1)$, for all j.*

Consider the following notation – for any $\gamma \in \mathbb{R}$, let $x_i(\gamma) \in [0, 1]$ satisfy, $v_i'(x_i(\gamma)) = \gamma$, and if no such argument exists let $x_i(\gamma) = 0$. In other words, x_i is the inverse function of v_i' when its input is within the proper range. Due to the concavity of v_i, $x_i(\gamma)$ is a decreasing function of γ, and so is $\sum_i x_i(\gamma)$. Also note that by the definition of α, $\sum_i x_i(\alpha) = \sum z_i = 1$.

We can now use Lemma 1 to provide conditions which are almost necessary and sufficient to determine the superiority between the two alternatives FULL and AON:

Theorem 2. *Given an economy with strictly concave valuations, the revenue from the AON mechanism is superior to that of FULL whenever $\sum_{i=1}^{n} x_i(v_2(1)) \leq 1$. The opposite holds whenever $\sum_{i=1}^{n} x_i(v_2(1)) \geq 2$.*

Note that, in fact, it is sufficient to require in the second part of Theorem 2 that for any j, $\sum_{i \neq j} x_i(v_2(1)) \geq 1$ in order to establish the superiority of FULL over AON. Also, no conclusions can be drawn from Theorem 2 whenever $\sum_{i=1}^{n} x_i(v_2(1)) \geq 1$ while for some player $\sum_{i \neq j} x_i(v_2(1)) < 1$.

3.3.1 Limits of quantitative advantage

We shall now attempt to analyze *how much* better one mechanism can be over the other and provide bounds on the revenue ratio.

Theorem 3. *Given an economy with strictly concave valuations, for any number of bidders, n, the revenue obtained under the FULL mechanism is at most $n - 1$ times that obtained under AON.*

In fact, $n-1$ is a tight bound as demonstrated in the following symmetric economy:

Example 2. *We define the function v_i as an arbitrary smooth concave function satisfying $v_i(1)=1$ and also*:

$$\left|v_i(x)-v_i^{\mathrm{PWL}}\right|<\frac{\varepsilon}{n(n-1)}$$

for all x, where v_i^{PWL} is the following piece-wise linear and continuous function:

$$\forall i: v_i^{\mathrm{PWL}}(x)=\begin{cases}(n-1)x & \text{if } x<\frac{1}{n-1}\\ 1 & \text{if } x\geq\frac{1}{n-1}\end{cases}$$

It can be shown that $R^{\mathrm{FULL}}>n-1-2\varepsilon$ (see appendix). Since $R^{\mathrm{AON}}=1$, it ensues that $\frac{R^{\mathrm{FULL}}}{R^{\mathrm{AON}}}>n-1-2\varepsilon$, where ε is arbitrarily small.

How much better can AON perform over FULL? The following example demonstrates that the ratio is unbounded:

Example 3. *Consider an infinite sequence of agents with valuations, $\{v_i\}_{i=1}^{\infty}$, that satisfy $v_i(x)=1$ for all $x>\frac{1}{2^{i+1}}$. For any subset of agents $1,\ldots,n$, $R^{\mathrm{AON}}=1$ while $R^{\mathrm{FULL}}=0$ (to see this, note that the maximal social welfare can be attained by allocating only half of the resource).*

3.4 Tiny economies

An immediate corollary is that in tiny economies, interpreted as economies with 2 buyers, AON outperforms FULL:

Corollary 2. *Given an economy with strictly concave valuations and 2 players, the revenue from the FULL mechanism is less or equal that of the AON mechanism.*

In fact, the revenue ratio in favor of AON is unbounded for tiny economies:

Example 4. *Consider 2 agents with the same valuation function $v(x)=min(2x,1)$. This function can be smoothed as done in Example 3. In this example, the revenue generated by AON is $R^{\mathrm{AON}}=1$ while that generated by FULL is $R^{\mathrm{FULL}}=2\varepsilon$, where ε is the smoothing factor. The ratio, $\frac{R^{\mathrm{AON}}}{R^{\mathrm{FULL}}}$, which depends on ε, can be made infinitely large.*

3.5 Replica economies

So far we studied general economies with n agents where all valuations were restricted to be concave or convex. We now turn to study a class of economies known in economics as "replica economics". A replica economy is an economy that is composed of some underlying fixed finite economy that is replicated.

Formally, fix a finite set of valuation functions, $\{v_i\}_{i=1}^n$, which we refer to as the base economy and consider the m-replica of this economy composed of mn agents with $v_{ij} = v_i$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.[2] The case where the base economy is composed of a single player is called a-*symmetric economy*.

It appears that for a large enough number of replicas, FULL extracts more revenue than AON.

Theorem 4. *Given any base economy, $\{v_i\}_{i=1}^n$, with strictly concave valuations, there exists some minimal number of replicas, M, such that FULL extracts more revenue than AON for any m-replica economy with $m > M$.*

3.5.1 Limits of quantitative advantage

How much better does FULL do over AON in replicas economies? Apparently, this depends on the derivative of the valuation functions at zero ($v_i'(0)$). Before we make the formal connection, we provide the following two examples which help to grasp the intuition behind it. In both examples the base economy's size is one.

In our first extreme example the revenue ratio is made arbitrarily close to one:

Example 5. *Consider a replica economy with a single valuation function, v, in each replica, satisfying $v'(x) \leq 1 + \varepsilon$ for all x and $v(1) = 1$ (so v is almost the diagonal). Recall that in the proof of Lemma 1 we established that the revenue from FULL satisfies $R^{\text{FULL}} \leq \alpha$ where $\alpha = v_{i'}(z_i)$. Therefore, $R^{\text{FULL}} \leq 1 + \varepsilon$. On the other hand the AON revenue is clearly $R^{\text{AON}} = 1$ and so $\frac{R^{\text{FULL}}}{R^{\text{AON}}} = 1 + \varepsilon$, which can be made arbitrarily close to 1.*

2 One can also extend the notion of a base economy to account for countably many buyers. Whereas, formally our results go through with this extension, we find this extension of little conceptual significance.

In the second extreme example the revenue ratio is unbounded and grows as the number of replicas increases:

Example 6. *Consider a replica economy with a single valuation function,* $v(x) = \sqrt{x}$*, in each replica. Note in particular that the derivative at zero is infinite. As in the previous example, it is clear that AON results in a revenue of 1* ($R^{\text{AON}} = 1$)*. On the other hand, the revenue from FULL can grow arbitrarily large with n (see appendix). Therefore, the revenue ratio satisfies* $\lim_{n\to\infty} \frac{R_n^{\text{FULL}}}{R_n^{\text{AON}}} = \infty$.

These two examples motivate the following observation:

Theorem 5. *Given a base economy with strictly concave valuations, the revenue from the FULL mechanism, in a corresponding replica economy,* R^{FULL}*, is no more than* $\max v_i'(0)$.

This allows us to observe that under the conditions of Theorem 5 the revenue ratio is bounded as follows:

Corollary 3. *Given an economy with strictly concave valuations, if every valuation function in the economy has a finite derivative at 0, then the revenue from the FULL mechanism is of the same order of magnitude as that of AON. Formally,* $\frac{R^{\text{FULL}}}{R^{\text{AON}}} = O(1)$.

Though FULL can sometimes be infinitely better than AON, this ratio is bounded by any linear function of the number of replicas.

Theorem 6. *Given a base economy with strictly concave valuations, the revenue from the FULL mechanism is bounded by the product of AON's revenue with any linear function of the number of replicas. In other words, it is of a lesser order of magnitude than* R^{AON} *times m (the number of replicas). Formally,* $\frac{R^{\text{FULL}}}{R^{\text{AON}}} = o(m)$.

On the other hand, we show below that R^{FULL} is not bounded by any sub-linear exponent:

Example 7. *Consider a replica economy with a base economy of one agent. Let the valuation function be* $v(x) = x^p$ *where* $0 < p < 1$*. Clearly* $R^{\text{AON}} = 1$ *in this case. We show in the appendix that for any* $0 < q < 1$ *there is a choice of p, for which a large enough M exists, such that for* $m > M$, $\frac{R^{\text{FULL}}}{R^{\text{AON}}} > (m-1)^q$*. In other words, for every sublinear function of m, a suitable replica economy of* x^p *valuation functions can be created, which is not bounded by that function.*

To summarize, we have shown that $\frac{R^{\text{FULL}}}{R^{\text{AON}}}$ is sublinear ($o(m)$), but that it is not limited by any sub-linear exponent of m. While this leaves us with a theoretical gap between our upper and lower bounds, it does shed light on the asymptotic behavior of replica economies.

3.6 Large symmetric economies

Note that in Theorem 4 we fix the base economy and study the implications of increasing the number of replicas. However, if we rig the base economy to the size of the economy (assuming the size of the base economy is fixed) we may obtain larger and larger replica economies where $R^{\text{FULL}} < R^{\text{AON}}$.

We demonstrate this for the classic case of-*symmetric economies* which are essentially economies where all agents have the same valuation. One can think of a symmetric economy as a replica economy with a singleton base economy ($m = 1$), however this may be misleading. Whereas in a replica economy we typically fix a base economy and study the properties of the economy as more replicas are added, the emphasis of our analysis of symmetric economies is on the implications of the structure of the single valuation function given a fixed number of buyers (which equals the number of replicas). In other words, we consider valuation functions that are sensitive to the number of agents in the economy (n) and use it as a parameter.

Example 8. *Consider a set of n players with the following valuation functions,* $v_i(x) = 2nx$ *for* $0 \leq x < \frac{1}{2n}$ *and* $v_i(x) = 1$ *for* $\frac{1}{2n} \leq x \leq 1$ *(one can smooth this function around* $\frac{1}{2n}$*). In this case* $v_2(1) = 1$*, and so* $\sum_{i=1}^{n} x_i(v_2(1)) = \sum_{i=1}^{n} x_i(1) = \sum_{i=1}^{n} \frac{1}{2n} = \frac{1}{2} < 1$ *and therefore, by Theorem 2, AON yields more revenue than FULL.*

In the following example, we show that not only can AON generate more revenues than FULL in the symmetric case but ratio between R^{FULL} and R^{AON} can approach zero arbitrarily fast, for increasing symmetric economies:

Example 9. *Let n be the number of buyers and set the valuation of each buyer to be* $v(x) = x^{-log_n\left(1-\frac{\varepsilon}{n(n-1)}\right)}$. *On the one hand* $R_n^{\text{AON}} = 1$ *and on the other hand* $R_n^{\text{AON}} < \varepsilon$ *(see appendix). Therefore,* $\frac{R^{\text{AON}}}{R^{\text{FULL}}}$ *can be arbitrarily large.*

As for the other directions. Indeed FULL can be more profitable than AON; however, the ration of the corresponding revenue cannot increase too fast as the size of the symmetric economy grows:

Theorem 7. *Given a symmetric economy with n buyers and strictly concave valuations, the revenue from the FULL mechanism is no more than $n-1$ times the revenue from the AON mechanism. Furthermore, this bound is tight.*

3.7 Summary of main results

Our quantitative results are summarized in Table 1, showing that the choice of selling a good in a divisible versus an indivisible way is dependent on the structure of the economy.

Table 1: Bounds for concave value functions. All bounds are tight except for $o(m)$, which is almost tight: for all $\varepsilon > 0$, there exist an $\omega(m^{1-\varepsilon})$ example. In other words, the actual bound is greater than every sublinear power function, and less than any linear function

	General economy	Tiny economy	Large replica economy	Large symmetric economy
2*$\frac{R^{\text{FULL}}}{R^{\text{AON}}}$	2*$\leq n-1$	2*≤ 1	2*$o(m)$	2*$\leq n-1$
2*$\frac{R^{\text{AON}}}{R^{\text{FULL}}}$	2*Unbounded	2*Unbounded	2* ≤ 1	2*Unbounded

In terms of *qualitative* results, we have provided the following:
1. We presented an almost complete characterization of when it is optimal to bundle a divisible and homogeneous good as an indivisible good.
2. We have shown that eliciting more information from the buyers is not always beneficial to the seller. In particular, it is never advantageous when there are only 2 buyers.
3. In sufficiently large replica economies ($m >> 1$) selling the resource as a fully divisible good (FULL) generates more revenues than bundling (AON).

4 Discussion

Motivated by the challenges of pricing computing infrastructure in the cloud, we provide a quantitative comparison for two natural schemes for selling a single homogeneous divisible good. We compare the classical VCG auction (which we refer to as the FULL mechanism) with the second price auction over the whole good (the AON mechanism). Other schemes for selling such a good may be relevant and could perform even better than these two but a full analysis of the optimal scheme is beyond the scope of this work. With that said, we dwell on two research directions which we think are important.

4.1 In between divisibility and indivisibility

The results we provided so far compare, in some sense, two extreme approaches to sell a divisible and homogeneous good. On one side we allow to divide the good in any possible way and on the other side we allow for no such flexibility. Clearly there are interim methods where some divisibility is allowed, without the full flexibility assumed under FULL. Consider, as an example, the following mechanism which we refer to as POINT mechanism. In POINT the seller uses a VCG mechanism but restricts buyers to provide a point valuation for the good. That is, buyers choose how much of the good they desire and what they are willing to pay for it. Formally, buyers report a pair in $M = [0, 1] \times \mathbb{R}_+$. Such a pair (z, y) is interpreted as the following step function: $v_i(x) = 0 \ \forall \ 0 \leq x < z$ and $v_i(x) = y \ \forall \ z \leq x \leq 1$.[3] With these function at hand the allocation and pricing scheme of FULL is implemented.

Analyzing the equilibrium outcome of POINT is typically complex as no dominant strategies necessarily exist. In fact, the natural solution concept to pursue is that of ex-post equilibria.[4] One such example of an ex-post equilibrium is where all buyers submit their true valuation for the whole good. This would of course result in similar consequences to the AON dominant strategy equilibrium outcome. Unfortunately, this is just one of many ex-post equilibria in this setting. Some of these equilibria could be inferior in terms of revenues, in comparison with AON, while others are simultaneously superior. This variance is demonstrated in the following examples:

Example 10. Superiority of AON over POINT: *There are two players, and each player has a concave valuation function that takes the value 1 for the whole resource, but is otherwise arbitrary. For each player to request half of the resource is an ex-post equilibrium. To observe this, note that if there exists a profitable deviation for one of the players then deviating to request 1 is also profitable. Assume, w.l.o.g. that for some functions v_1, v_2 it is profitable for player 1 to deviate. This implies that*:

$$v_1(1) - v_2(0.5) > v_1(0.5)$$

Due to concavity $v_i(0.5) > 0.5$, and so:

$$v_1(1) - v_2(0.5) \leq 1 - 0.5 = 0.5 \leq v_1(0.5)$$

3 Note that POINT captures the scheme adopted by Amazon in their recent release of the AWS spot market. See: www.infoworld.com/d/cloud-computing/amazon-unveils-ebay-style-auction-service-cloud-computing-161.

4 Roughly speaking, a tuple of prescribed agents' behaviors (bids) is an ex-post equilibrium if no agent can benefit from deviating, even after learning the others' bid.

contradicting the previous inequality. Now note that the revenue generated in this ex-post equilibrium is zero, compared with the revenue of 1 with AON.

One can observe that, in fact, any pair of requests of the form $(x, 1-x)$ is an ex-post equilibrium which generates no revenues at all. The following example, however, demonstrates the possibility of generating more revenue in an ex-post equilibrium:

Example 11. Superiority of POINT over AON: *There are 3 players, with the same valuation functions as in the previous example. The vector of claims where each players claims 0.5 of the resource is an ex-post equilibrium. Assume, w.l.o.g., that players 1 and 2 are allocated the resource, then the revenue is*:

$$2 \cdot v_3(0.5) > 1$$

In this setting having player 1 request the whole good while players 2 and 3 request nothing is also an ex-post equilibrium which generates no revenue.

4.2 Reserve price

It is well known that optimal auctions for indivisible goods involve setting reserve prices for the goods (Myerson, 1981). Thus, an analysis of optimal mechanisms for the divisible case is not complete if such reserve prices are ignored. Nevertheless we chose to relegate this analysis to future research for various reasons. First and foremost an analysis of optimal auctions in general and revenue in the presence of reserve prices in particular must account for the prior distribution on agents' valuations, a component of the model that was redundant for the analysis we provided thus far. Second, whereas our results which apply to a non-Bayesian setting carry over to the complete information model as well, this is not true for auctions with reserve prices. In the latter case, the auctioneer can easily exploit the information on valuations and set such reserve prices to extract the full surplus.

Let us add, at this point, that the notion of a reserve price may not be well defined and one can think of various natural forms for formalizing this notion. One such form is to set a reserve price for the good as a whole, with a pro-rated computation for any part thereof. Another option is to have a fixed reserve price, independent of size. Finally, the most general model is to represent the reserve price as an arbitrary function of the size of the good. This subsumes both previous models, however at a potential cost of substantial communication complexity.

Appendix: Theorem Proofs

Theorem 1 If all the functions, v_i, are convex then both in AON and in FULL player 1 is allocated the whole good and he pays the valuation of player 2 for the whole good, namely $v_2(1)$.

Proof: From Proposition 1 we know that either $z_1 = 1$ or $\sum_{\{i:z_i>0\}} v_i''(z_i) < 0$. The latter is impossible as the functions v_i'' are non-negative and so it must be the case that player 1 is allocated the whole good. Reiterating this for the set of players without player 1 we know that $z_2^1 = 1$ and the theorem follows. □

Lemma 1 Given an economy with strictly concave valuations, the revenue extracted in the FULL mechanism, R^{FULL}, is bounded as follows:

$$\sum_{j=1}^{n} \beta_j z_j \leq R^{\text{FULL}} \leq \alpha$$

Proof: The valuation functions are concave, therefore:

$$\begin{aligned}
R^{\text{FULL}} &= \sum_{j=1}^{n} \sum_{i \neq j} \left(v_i(z_i^j) - v_i(z_i) \right) \\
&= \sum_{j=1}^{n} \sum_{i \neq j} \frac{v_i(z_i^j) - v_i(z_i)}{z_i^j - z_i} \times \left(z_i^j - z_i \right) \\
&\leq \sum_{j=1}^{n} \sum_{i \neq j} v_{i'}(z_i) \times \left(z_i^j - z_i \right) = \sum_{j=1}^{n} \sum_{i \neq j} \alpha \times \left(z_i^j - z_i \right) \\
&= \alpha \sum_{j=1}^{n} \sum_{i \neq j} (z_i^j - z_i) = \alpha
\end{aligned}$$

and the right inequality holds. Similarly,

$$\begin{aligned}
R^{\text{FULL}} &= \sum_{j=1}^{n} \sum_{i \neq j} \left(v_i(z_i^j) - v_i(z_i) \right) \\
&= \sum_{j=1}^{n} \sum_{i \neq j} \frac{v_i(z_i^j) - v_i(z_i)}{z_i^j - z_i} \times \left(z_i^j - z_i \right)
\end{aligned}$$

$$\geq \sum_{j=1}^{n} \sum_{i \neq j} v_{i'}(z_i^j) \times \left(z_i^j - z_i\right) = \sum_{j=1}^{n} \sum_{i \neq j} \beta_j \times \left(z_i^j - z_i\right)$$

$$= \sum_{j=1}^{n} \beta_j \left(\sum_{i \neq j} z_i^j - \sum_{i \neq j} z_i \right) = \sum_{j=1}^{n} \beta_j (1 - (1 - z_j)) = \sum_{j=1}^{n} \beta_j z_j$$

and the left inequality holds. □

Corollary 1 Given an economy with strictly concave valuations, the revenue from the AON mechanism is superior to that of the FULL mechanism whenever $\alpha \leq v_2(1)$. The opposite holds if $\beta_j \geq v_2(1)$, for all j.

Proof:
1. $R^{\text{FULL}} \leq \alpha \leq v_2(1) = R^{\text{AON}}$.
2. $R^{\text{FULL}} \geq \sum_{j=1}^{n} \beta_j z_j \geq \sum_{j=1}^{n} v_2(1) z_j = R^{\text{AON}}$. □

Theorem 2 Given an economy with strictly concave valuations, the revenue from the AON mechanism is superior to that of FULL whenever $\sum_{i=1}^{n} x_i(v_2(1)) \leq 1$. The opposite holds whenever $\sum_{i=1}^{n} x_i(v_2(1)) \geq 2$.

Proof: In case $\sum_{i=1}^{n} x_i(v_2(1)) \leq 1$, since $x_i(\gamma)$ are decreasing in γ we deduce that $\alpha \leq v_2(1)$ and so the first part of the theorem follows from the first part of the previous corollary. In case $\sum_{i=1}^{n} x_i(v_2(1)) \geq 2$, then for any j, $\sum_{i \neq j} x_i(v_2(1)) \geq 1$. Thus, it must be the case that $\forall j \;\; \beta_j \geq v_2(1)$. The proof follows from part two of the previous corollary. □

Theorem 3 Given an economy with strictly concave valuations, for any number of bidders, n, the revenue obtained under the FULL mechanism is at most $n-1$ times that obtained under AON.

Proof: We denote by $R(i)$ the payment of buyer i under FULL. Therefore, $R_n^{\text{FULL}} = \sum_{i=1}^{n} R(i)$. As the VCG mechanism is individually rational $R_i \leq v_i(z_i)$. In addition $R(1) = \sum_{i=2}^{n} \left[v_i(z_i^1) - v_i(z_i)\right]$. Therefore:

$$\begin{aligned} R_n^{\text{FULL}} &= R(1) + \sum_{i=2}^{n} R(i) \\ &\leq \sum_{i=2}^{n} \left[v_i(z_i^1) - v_i(z_i)\right] + \sum_{i=2}^{n} v_i(z_i) \end{aligned}$$

$$= \sum_{i=2}^{n} v_i(z_i^1) \leq \sum_{i=2}^{n} v_i(1) \leq \sum_{i=2}^{n} v_2(1)$$

$$= (n-1) \cdot v_2(1) = (n-1) R_n^{\mathrm{AON}}$$

which entails the desired result. □

Example 2 We define the function v_i as an arbitrary smooth concave function satisfying $v_i(1) = 1$ and also:

$$\left| v_i(x) - v_i^{PWL} \right| < \frac{\varepsilon}{n(n-1)}$$

for all x, where v_i^{PWL} is the following piece-wise linear and continuous function:

$$\forall i : v_i^{\mathrm{PWL}}(x) = \begin{cases} (n-1)x & \textit{if } x < \frac{1}{n-1} \\ 1 & \textit{if } x \geq \frac{1}{n-1} \end{cases}$$

$$v_i\left(\frac{1}{n-1}\right) - v_i\left(\frac{1}{n}\right) > 1 - \frac{n-1}{n} - 2\frac{\varepsilon}{n(n-1)}$$

$$= \frac{1}{n} - 2\frac{\varepsilon}{n(n-1)}$$

and so:

$$R^{\mathrm{FULL}} = n \cdot (n-1) \cdot \left(v_i\left(\frac{1}{n-1}\right) - v_i\left(\frac{1}{n}\right) \right)$$

$$> n \cdot (n-1) \cdot \left(\frac{1}{n} - 2\frac{\varepsilon}{n(n-1)} \right) = n - 1 - 2\varepsilon$$

On the other hand, $R^{\mathrm{AON}} = 1$ and so $\frac{R^{\mathrm{FULL}}}{R^{\mathrm{AON}}} > n - 1 - 2\varepsilon$, where ε is arbitrarily small.

Corollary 2 Given an economy with strictly concave valuations and 2 players, the revenue from the FULL mechanism is less or equal that of the AON mechanism.

Proof: The proof follows from setting $n = 2$ in Theorem 3. □

Theorem 4 Given any base economy, $\{v_i\}_{i=1}^{n}$, with strictly concave valuations, there exists some minimal number of replicas, M, such that FULL extracts more revenue than AON for any m-replica economy with $m > M$.

Proof: Let $\bar{x}_i$ denote $x_i(v_2(1))$. Due to concavity, it must be the case that $\bar{x}_2 = x_2(v_2(1)) > 0$ and consequently $\sum_{i=1}^{m} \bar{x}_i > 0$. Therefore, $N = \frac{2}{\sum_{i=1}^{m} \bar{x}_i}$ is well defined. For any $n > N$, $\sum_{i=1,j=1}^{n,m} x_{ij}(v_2(1)) = \sum_{i=1}^{n} \sum_{i=1}^{m} \bar{x}_i > N \sum_{j=1}^{m} \bar{x}_i = 2$ and the result follows from the second part of Theorem 2. □

Example 6 Consider a replica economy with a single valuation function, $v(x) = \sqrt{x}$, in each replica. Note in particular that the derivative at zero is infinite. As in the previous example, it is clear that AON results in a revenue of 1 ($R^{\text{AON}} = 1$). On the other hand the revenue from FULL requires some computation:

$$R^{\text{FULL}} = n \cdot (n-1) \cdot \left(v\left(\frac{1}{n-1}\right) - v\left(\frac{1}{n}\right) \right) =$$

$$n \cdot (n-1) \left(\sqrt{\frac{1}{n-1}} - \sqrt{\frac{1}{n}} \right) \rightarrow_{n \to \infty} \infty$$

Therefore, the revenue ratio satisfies $lim_{n \to \infty} \frac{R_n^{\text{FULL}}}{R_n^{\text{AON}}} = \infty$.

Theorem 5 Given a base economy with strictly concave valuations, the revenue from the FULL mechanism, in a corresponding replica economy, R^{FULL}, is no more than $max\ v_i'(0)$.

Proof: Recall that $R^{\text{FULL}} \leq v_i'(z_i)$ for every player i with a positive allocation (Lemma 1). Since v_i' is a decreasing function, we can infer that $R^{\text{FULL}} \leq max\ v_i'(0)$. □

Theorem 6 Given a base economy with strictly concave valuations, the revenue from the FULL mechanism is bounded by the product of AON's revenue with any linear function of the number of replicas. In other words, it is of a lesser order of magnitude than $m \cdot R^{\text{AON}}$. Formally, $\frac{R^{\text{FULL}}}{R^{\text{AON}}} = o(m)$.

Proof: Assume without loss of generality that $v_1(1) = 1$ and that $m \geq 2$. It follows that $R^{\text{AON}} = 1$, and therefore proving $R^{\text{FULL}} = o(m)$ will suffice.

Each agent pays his own bid at most. Since there are m replicas of each agent type, each agent is allocated a maximum of $\frac{1}{m}$. Otherwise, some agent is allocated more than $\frac{1}{m}$ and so one of its clones must be allocated less than that. The sum of the utilities of these 2 agents can be increased by equating their portions (due to the concavity of the valuation function), while not changing any allocation of the other agents. Consequently, the social welfare would increase, therefore contradicting the assumption that an optimal allocation has been chosen. Therefore, every agent is allocated $\frac{1}{m}$ of the good at most.

$$R^{\text{FULL}} \le m \sum_{i=1}^{n} v_i(z_i) \le m \sum_{i=1}^{n} v_i\left(\frac{1}{m}\right)$$

Let $v^*(x) = max\ v_i(x)$. This gives us the following bound:

$$R^{\text{FULL}} \le m \cdot n \cdot v^*\left(\frac{1}{m}\right)$$

Clearly v^* is continuous and in addition $v^*(0) = 0$ and so for any $\varepsilon > 0$ there exists some large enough M such that for any $m > M$, $n \cdot v^*\left(\frac{1}{m}\right) < \varepsilon$, which in turn implies $R^{\text{FULL}} < \varepsilon m$, and so $R^{\text{FULL}} = o(m)$. □

Example 7 Consider a replica economy with a base economy of one agent. Let the valuation function be $v(x) = x^p$ where $0 < p < 1$. Clearly $R^{\text{AON}} = 1$ in this case. We compute a lower bound on:

$$R^{\text{FULL}} = \frac{v\left(\frac{1}{m-1}\right) - v\left(\frac{1}{m}\right)}{\frac{1}{m-1} - \frac{1}{m}} \ge v'\left(\frac{1}{m-1}\right) = p(m-1)^{1-p}$$

For any $0 < q < 1$ we can choose $p = 1 - q - \varepsilon$. For a large enough m, we note that $(m-1)^{\varepsilon} > \frac{1}{p}$, ultimately yielding:

$$(m-1)^q = (m-1)^{1-p-\varepsilon} < p(m-1)^{1-p} \le R^{\text{FULL}}$$

Therefore, for any $0 < q < 1$ there exists a large enough M such that for $m > M$, $\frac{R^{\text{FULL}}}{R^{\text{AON}}} > (m-1)^q$.

Example 9 Let n be the number of buyers and set the valuation of each buyer to be $v(x) = x^{-\log_n\left(1 - \frac{\varepsilon}{n(n-1)}\right)}$. On the one hand $R_n^{\text{AON}} = 1$ and on the other hand:

$$\begin{aligned} R_n^{\text{FULL}} &= n \cdot (n-1) \cdot \left(v_i\left(\frac{1}{n-1}\right) - v_i\left(\frac{1}{n}\right)\right) \\ &< n \cdot (n-1) \cdot \left(v_i(1) - v_i\left(\frac{1}{n}\right)\right) \\ &= n \cdot (n-1) \cdot \left(1 - \left(\frac{1}{n}\right)^{-\log_n\left(1 - \frac{\varepsilon}{n(n-1)}\right)}\right) = \varepsilon \end{aligned}$$

and therefore $\frac{R^{\text{AON}}}{R^{\text{FULL}}}$ can be arbitrarily large.

Theorem 7 Given a symmetric economy with n buyers and strictly concave valuations, the revenue from the FULL mechanism is no more than $n-1$ times the revenue from the AON mechanism. Furthermore, this bound is tight.

Proof: Follows from Theorem 3 and Example 2. □

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References

Adams, W. J., and J. L. Yellen. 1976. "Commodity Bundling and the Burden of Monopoly." *The Quarterly Journal of Economics* 90(3):475–98. MIT Press.

Ben-Yehuda, O. A., M. Ben-Yehuda, A. Schuster, and D. Tsafrir. 2012. "The Resource-as-a-Service (Raas) Cloud." In Proceedings of the 4th USENIX conference on Hot Topics in Cloud Computing, USENIX Association, 12–12.

Brânzei, S., and P. Bro Miltersen. 2013. "Equilibrium Analysis in Cake Cutting." In *International Conference on Autonomous Agents and Multi-Agent Systems, AAMAS, 13, Saint Paul, MN, USA, May 6–10 2013*, edited by M.L. Gini, O. Shehory, T. Ito, and C.M. Jonker, 327–34. Saint Paul, MN: IFAAMAS. ISBN: 978-1-4503-1993-5.

Clarke, E. H. 1971. "Multipart Pricing of Public Goods." *Public Choice* 11:17–33.

Feige, U., and M. Tennenholtz. 2011. "Mechanism Design with Uncertain Inputs: (to Err Is Human, to Forgive Divine)." In *Proceedings of the 43rd ACM Symposium on Theory of Computing, STOC 2011, San Jose, CA, USA, June 6–8 2011*, edited by L. Fortnow and S.P. Vadhan, 549–58. San Jose, CA: ACM. ISBN: 978-1-4503-0691-1

Ghodsi, A., M. Zaharia, B. Hindman, A. Konwinski, S. Shenker, and I. Stoica. 2011. "Dominant Resource Fairness: Fair Allocation of Multiple Resource Types." In Proceedings of the 8th USENIX Conference on Networked Systems Design and Implementation, NSDI'11, Berkeley, CA, USENIX Association, 24–24. http://dl.acm.org/citation.cfm?id=1972457.1972490.

Ghosh, A., H. Nazerzadeh, and M. Sundararajan. 2007. "Computing Optimal Bundles for Sponsored Search." In *Internet and Network Economics, Third International Workshop, WINE 2007, San Diego, CA, USA, December 12–14 2007, Proceedings*, edited by X. Deng and F.C. Graham, 576–83. Lecture Notes in Computer Science 4858. Springer. ISBN: 978-3-540-77104-3

Groves, T. 1973. "Incentives in Teams." *Econometrica* 41(4):617–31.

Holzman, R., N. Kfir-Dahav, D. Monderer, and M. Tennenholtz. 2004. "Bundling Equilibrium in Combinatorial Auctions." *Games and Economic Behavior* 47:104–23.

Holzman, R., and D. Monderer. 2004. "Characterization of Ex Post Equilibrium in the VCG Combinatorial Auctions." *Games and Economic Behavior* 47:87–103.

Jehiel, P., M. Meyer-Ter-Vehn, and B. Moldovanu. 2007. "Mixed Bundling Auctions." *Journal of Economic Theory* 134:494–512.

Kelly, F. 1997. "Charging and Rate Control for Elastic Traffic." *European Transactions on Telecommunications* 8:33–37.

Lazar, A. A., and N. Semret. 1999. "Design and Analysis of the Progressive Second Price Auction for Network Bandwidth Sharing." *Telecommunication Systems* 20:255–63.

Maillé, P., and B. Tuffin. 2004. "Multibid Auctions for Bandwidth Allocation in Communication Networks." In *INFOCOM 2004*, Volume 1, IEEE.

Myerson, R. B. 1981. "Optimal Auction Design." *Mathematics of Operations Research* 6:58–73.

Palfrey, T. R. 1983. "Bundling Decisions by a Multiproduct Monopolist with Incomplete Information." *Econometrica* 51(2):463–83.

Pettey, C., and R. van der Meulen. 2012. "Gartner Says Worldwide Cloud Services Market to Surpass $109 Billion in 2012." http://www.gartner.com/newsroom/id/2163616

Sanghavi, S., and B. Hajek. 2004. "Optimal Allocation of a Divisible Good to Strategic Buyers." In *CDC 2004*, Volume 3, 2748–53. Bahamas: IEEE.

Smirnov, V., and A. Wait. 2011. "Bundling in Auctions with Non-Linear Valuations." *Australian Economic Papers* 50:17–26.

Vickrey, W. 1961. "Counterspeculation, Auctions, and Competitive Sealed Tenders." *The Journal of Finance* 16:8–37.